Phase Space Exchange in Thick Wedge Absorbers for Ionization Cooling

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Abstract

The problem of phase space exchange in wedge absorbers with ionization cooling is discussed. The wedge absorber exchanges transverse and longitudinal phase space by introducing a position-dependent energy loss.   In this paper we note that the wedges used with ionization cooling are relatively thick, so that single wedges cause relatively large changes in beam phase space. Calculation methods adapted to such "thick wedge" cases are presented, and beam phase-space transformations through such wedges are discussed.

Introduction

Recent innovations have intensified research on the possibility of a high-intensity high-luminosity $\mu^+\mu^-$ collider.[1,2] In this collider concept ionization cooling is used to increase the phase space density of the beam by a factor of $\sim 10^6$. In ionization cooling the beam passes through an absorber, in which it loses momentum parallel to its motion, followed by reacceleration in which it regains only longitudinal momentum. The transverse components of the motion are effectively reduced and therefore cooled in each step, but the procedure is inefficient in cooling longitudinal phase space. However, phase-space exchange between longitudinal and transverse phase space is possible by passing the beam through a wedge absorber (in which absorber thickness or density depends on transverse position), and the degree of exchange is controllable by changing the dispersion at the absorber. The scenario for phase space cooling alternates energy-loss transverse cooling with transverse-longitudinal exchanges, in order to obtain cooling in all 6-D phase space dimensions.[3]

In previous discussions of ionization cooling,[4, 5, 6] this phase-space exchange has been included in the differential equations of cooling, following a similar exchange occuring in synchrotron radiation damping.[7] That treatment implies a small exchange per integration step. In ionization cooling relatively large exchanges can occur in a single wedge, and a formalism more appropriate to these large exchanges in a single step is desirable, particularly in the development of explicit designs. In this paper, we separate the cooling and heating aspects of the ionization from the phase-space exchange, and consider the exchange with dispersion as a linear transformation. Based on the betatron function formalism[8] and following a related application by J. Peterson [9], we present simplified formulae for the amount of phase space exchange and the related transformations of transverse and longitudinal phase-space parameters (emittances, betatron functions, dispersion, momentum width, etc.). We apply these to particular examples, and deduce optimized conditions and cases for phase-space exchange.

Formalism

Figure 1 shows a stylized view of the passage of beam with dispersion through an absorber. The key parameters are the beam properties entering and exiting the wedge, as well as the properties of the wedge itself. The beam properties entering the wedge are the beam energy and momentum E_0, p_0, the beam momentum width $\Delta p_0 = \Delta E_0/(v/c)$ and the beam transverse emittance ε_0. We use the relative momentum width defined by $\delta_0 = \Delta p_0/p_0$. The beam transport properties are given by the betatron function β_0, and the dispersion η_0. To simplify discussion, the beam is focussed to a betatron and dispersion waist at the wedge: $\beta_0' $, $\eta_0' = 0$. (This is an optimal choice and permits us to avoid any changes in β', η' in the wedge.) The beam properties after the wedge are represented by the same symbols with the subscript 1: E_1, p_1, δ_1, ε_1, β_1, η_1. The wedge is represented by its relative effect on the momentum offsets δ of particles within the bunch at position x:

$$\frac{\Delta p}{p} = \delta \rightarrow \delta - \frac{(dp/ds)\tan\theta}{p}x = \delta - \delta'x$$

dp/ds is the momentum loss rate in the material (dp/ds = β^{-1}dE/ds), which is dependent on the material and the particle energy. For beryllium with beam at the minimimum-ionizing energy, dp/ds is ~3.0 MeV/c/cm. x tan$q$ is the wedge thickness at transverse position x (relative to the central orbit at x=0), and we have introduced the symbol $\delta' = $ dp/ds tan$q$/p to indicate the change of δ with x. At typical values for ionizing cooling wedges ($\delta = 0.01$, p =300 MeV/c, tan$q$ up to 1, x = 1cm), changes in δ are relatively large, so a thick wedge treatment is desirable.

Note that changes in the transverse beam only occur in the coordinate with wedge thickness variation (horizontal for horizontal bends and wedges). The perpendicular beam projection is unchanged. Horizontal and vertical effects can be balanced by including both horizontal and vertical bend and wedge sections.

In practical applications, we would like to constrain the wedge geometry so that the beam entrance and exit angles are not too steep; a maximum value of θ of ~45° would seem reasonable (tan $\theta = 1$). The angles could be increased if a symmetric wedge of angle ϕ is used (see figure 2); the two wedges are equivalent if tanθ =2 tan(ϕ/2). The symmetric wedge is also practically constrained to $\phi < 45°$, but this extends the range of tan θ to ~2.

In first order, we ignore the central beam energy loss, as well as multiple scattering and energy straggling in the absorber, and consider only the effects of δ' and η_0. In that case, δ' and η_0 can be represented as linear transformations in the x-δ phase space projections and the transformations are phase-area preserving. Thus the dispersion can be represented by the

matrix:

$$\mathbf{M}_\eta = \begin{bmatrix} 1 & \eta_0 \\ 0 & 1 \end{bmatrix} \quad , \text{ since } x \Rightarrow x + \eta_0\delta$$

and the wedge can be represented by the matrix: $\quad \mathbf{M}_\delta = \begin{bmatrix} 1 & 0 \\ -\delta' & 1 \end{bmatrix} \quad$, so that the dispersion +

wedge becomes: $\quad \mathbf{M}_{\eta\delta} = \begin{bmatrix} 1 & \eta_0 \\ -\delta' & 1-\delta'\eta_0 \end{bmatrix}.$

Under the general assumption of smoothly-populated beam distributions, we can represent the x-δ beam distribution as a phase space ellipse :

$$g_0 x^2 + b_0 \delta^2 = \sigma_0 \delta_0$$

where σ_0 is the initial zero-η beam size (related to transverse emittance and betatron function by $\sigma_0 = (\varepsilon_0\beta_0)^{1/2}$).   Also $g_0 = \delta_0/\sigma_0$, and $b_0 = \sigma_0/\delta_0$. After the dispersion plus wedge the beam is within another equal-area phase-space ellipse, given by:

$$g_1 x^2 + 2a_1 x\delta + b_1 \delta^2 = \sigma_0 \delta_0$$

where the revised ellipse parameters are found from the old ones and the transfer matrix by standard betatron function transport techniques:[8]

$$b_1 = b_0 + (\eta_0)^2 g_0$$
$$a_1 = \delta' b_0 - \eta_0(1-\delta'\eta_0)g_0$$
$$g_1 = \delta'^2 b_0 + (1-\delta'\eta_0)^2 g_0$$

From these revised parameters, the rotated beam parameters can be deduced.

The energy width is changed to:

$$\delta_1 = \sqrt{g_1\sigma_0\delta_0} = \delta_0 \left[(1-\eta_0\delta')^2 + \frac{\delta'^2\sigma^2}{\delta_0^2} \right]^{1/2}.$$

In the transformations the bunch length is unchanged. The longitudinal emittance, the area of the beam in longitudinal phase-space (energy-width $\times$ bunch-length), has therefore changed simply by the ratio of energy-widths, which means that the longitudinal emittance has changed by the factor δ_1/δ_0.

From emittance conservation, the transverse emittance has changed by the inverse of this factor:

$$\varepsilon_1 = \varepsilon_0 \left[(1 - \eta_0 \delta')^2 + \frac{\delta'^2 \sigma^2}{\delta_0^2} \right]^{-1/2} .$$

This same factor can also be obtained by noting that the transverse beam size can be written as $\sigma_1 = (b_1 \sigma_0 \delta_0)^{1/2}$, calculating b_1 and noting that transverse velocities are unchanged. The emittance, which is simply the product of size and velocity, can then be obtained from $\varepsilon_1/\varepsilon_0 = \sigma_1/\sigma_0$.

The x-δ phase space ellipse equation can be rewritten into the form:

$$g_1 \left(x + \frac{a_1}{g_1} \delta \right)^2 + \frac{\delta^2}{g_1} = \sigma_0 \delta_0 ,$$

where we have used $b_1 g_1 = 1 + a_1^2$. From this form we can see that the new value of the dispersion is: $\eta_1 = -a_1/g_1$, which can be written as

$$\eta_1 = -\frac{a_1}{g_1} = \frac{\eta_0 (1 - \eta_0 \delta') - \delta' \frac{\sigma^2}{\delta_0^2}}{(1 - \eta_0 \delta')^2 + \delta'^2 \frac{\sigma^2}{\delta_0^2}}$$

The rms uncorrelated beam size (dispersion-removed) has also changed to:

$$\sigma_1^2 = \left\langle (x - \eta_1 \delta)^2 \right\rangle = \sigma_0 \delta_0 / g_1 ,$$

which means: $\sigma_1 = \sigma_0 \left[(1 - \eta_0 \delta')^2 + \frac{\delta'^2 \sigma_0^2}{\delta_0^2} \right]^{-1/2}$. Combining this with our expression for the transverse emittance, and using $\sigma_1^2 = \varepsilon_1 \beta_1$, we find that the transverse betatron function has also been changed in the same proportion:

$$\beta_1 = \beta_0 \left[(1 - \eta_0 \delta')^2 + \frac{\delta'^2 \sigma^2}{\delta_0^2} \right]^{-1/2} .$$

Note that the change in betatron functions (β_1, η_1) implies that the following optics should be correspondingly rematched.

Particular Cases

We have, in the above equations, completely re-characterized the phase-rotated beam. We can apply the above results to particular cases, some of which are of great practical importance, and which can simplify the cooling process.

A. Thin Wedge

In the limit where $\delta' \to 0$ (which means small tilt angle $\boldsymbol{q}$ and/or large p and/or small dp/ds), the emittance change per wedge is small. In that limit:

$$\varepsilon_1 \to \varepsilon_0(1 + \eta_0\delta')$$

$$\delta_1 \to \delta_0(1 - \eta_0\delta')$$

$$\eta_1 \to \eta_0(1 + \eta_0\delta') - \delta'\frac{\sigma^2}{\delta_0{}^2}$$

$$\beta_1 \to \beta_0(1 + \eta_0\delta')$$

This is the same result previously derived in reference 4 for the case of continuous small changes. Note that a dispersion develops from a wedge absorber even in the absence of initial dispersion, and a second-order emittance exchange develops from integration of that.

B. Maximum Exchange

A maximal exchange effect occurs when $\delta' = 1/\eta_0$, and many terms in the equations are cancelled out. The relationship between initial and resulting parameters are greatly simplified:

$$\delta_1 = \delta_0\frac{\sigma_0}{\eta_0\delta_0}, \varepsilon_1 = \varepsilon_0\frac{\eta_0\delta_0}{\sigma_0}, \beta_1 = \beta_0\frac{\eta_0\delta_0}{\sigma_0}, \eta_1 = -\eta_0.$$

In this case the dispersion beam size $\eta\delta$ and the emittance beam size σ are simply exchanged.

In this "maximal exchange" condition, the emittances change simply as the ratio of the momentum beam size $\eta_0\delta_0$ to the initial (emittance) beam size σ_0, obtaining a large exchange when that ratio is large. We also note that dispersion actually changes sign in the absorber. The equations also show a large inverse exchange for small η. However this inverse exchange may require a large δ', and δ' is limited by our geometrical constraint of tan $\theta < {\sim}1$. (The case of large δ' with $\eta=0$ is discussed below.)

C. Dispersion-matched exchange

One can similarly choose δ' and η_0 to obtain dispersion-cancelling; that is, a solution in which the resulting dispersion η_1 is zero. This can be obtained when $a_1 \to 0$, which implies that

$$\delta' = \frac{1}{\eta_0\left[1 + \frac{\sigma_0{}^2}{\eta_0{}^2\delta_0{}^2}\right]}$$, a similar but somewhat reduced case from maximal exchange.

In this case, somewhat simplified emittance exchange fomulae are obtained:

$$\delta_1 = \frac{\delta_0}{\sqrt{1 + \frac{\eta_0{}^2\delta_0{}^2}{\sigma_0{}^2}}}, \varepsilon_1 = \varepsilon_0\sqrt{1 + \frac{\eta_0{}^2\delta_0{}^2}{\sigma_0{}^2}}, \beta_1 = \beta_0\sqrt{1 + \frac{\eta_0{}^2\delta_0{}^2}{\sigma_0{}^2}}, \eta_1 = 0.$$

This case is particularly useful, because the cancelling of η implies that dispersion matching after the wedge is not necessary. This greatly simplifies the optics. As in the previous case the degree of exchange possible depends on the ratio of the momentum beam size $\eta_0\delta_0$ to initial (emittance) beam size σ_0, obtaining a large exchange when that ratio is large.

D. Wedge only.

As previously discussed, passing the beam through a wedge absorber without an initial dispersion introduces a dispersion in the beam. In a "thick" absorber that dispersion generates an emittance exchange, and this increases the longitudinal emittance and energy spread while decreasing the transverse emittance. The equations for emittance change and final dispersion are :

$$\delta_1 = \delta_0\sqrt{1+\frac{\delta'^2\sigma^2}{\delta_0{}^2}} , \varepsilon_1 = \frac{\varepsilon_0}{\sqrt{1+\frac{\delta'^2\sigma^2}{\delta_0{}^2}}} , \beta_1 = \frac{\beta_0}{\sqrt{1+\frac{\delta'^2\sigma^2}{\delta_0{}^2}}} , \eta_1 = -\frac{\delta'\sigma^2}{1+\delta'^2\sigma^2} .$$

This is also a very useful case, since it does not require dispersion matching into the wedge.

Integration with beam cooling and heating processes

In the linearized model, the mean beam energy is assumed to remain constant and no scattering is included. However the wedges are sufficiently thick to have significant energy loss and with that some emittance cooling, multiple scattering, and energy straggling. These effects should be included in a full discussion.

The differential equation for transverse cooling is:

$$\frac{d\varepsilon_N}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N + \frac{\beta_\perp E_s^2}{2\beta^3 m_\mu c^2 L_R E}$$

where the first term is the frictional cooling effect and the second is the multiple scattering heating term. Here L_R is the material radiation length, $\beta_\perp$ is the betatron function, and E_s is the characteristic scattering energy (~14 MeV). In a first approximation we can estimate the effect by simply multiplying by the thickness of the wedge Δz at the beam center. The required thickness is set by the requirement that the wedge cover at least $\pm 2\sigma$ in momentum and position, and the thickness goes to zero at $x = -\Delta z /\tan\theta$. This means $\Delta z > 2\eta\delta_0 \tan\theta$, and $\Delta z > 2\sigma_0 \tan\theta$ are required. Therefore:

$$\Delta\varepsilon_N\big|_{material} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N\Delta z + \frac{\beta_\perp E_s^2}{2\beta^3 m_\mu c^2 L_R E}\Delta z$$

and a choice of Δz as the geometric sum of $2\eta\delta_0 \tan\theta$ and $2\sigma_0 \tan\theta$ is reasonable.

 Similarly the longitudinal cooling and heating equation is:

$$\frac{d\sigma_E^2}{ds} = -2 \frac{\partial \frac{dE}{ds}}{\partial E} \sigma_E^2 + 4\pi \left(r_e m_e c^2 \right)^2 n_e \gamma^2 \left(1 - \frac{\beta^2}{2} \right)$$

the first term is the frictional cooling (or heating) term, which is naturally small, except at low-energies ($E_\mu < 100$ MeV) where it heats the beam. The second term is the energy-straggling term, where n_e is the material electron density. From the above discussion, we estimate for moderate to high μ energies that

$$\Delta\delta^2 \big|_{material} \cong +4\pi \left(r_e m_e c^2 \right)^2 \frac{n_e}{\beta^4 (m_\mu c^2)^2} \left(1 - \frac{\beta^2}{2} \right) \Delta z$$

These effects are included in our discussions of examples. In optimal examples these effects should be relatively small.

Numerical examples and simulation results

In developing a possible scenario for the assembly of high-intensity bunches for a possible μ^+-μ^- collider, a beam cooling and compression scenario is being developed.[3] In that scenario a beam with a very large emittance, including large energy spread, is cooled by many passages (20—100) through material absorbers followed by rf reacceleration sections. That cooling sequence includes many wedge absorbers for exchange of transverse and longitudinal emittance. The cooling sequence begins with wedge absorbers to reduce the energy spread, followed by ~20 steps of transverse energy cooling intermixed with wedge exchangers. In the final steps, a relatively small longitudinal emittance with small energy-spread is wedge-exchanged to obtain a greatly reduced transverse emittance. In the full-sequence the transverse normalized emittances ($\varepsilon_N = \varepsilon_\perp \beta\gamma$) are reduced by ~ two orders of magnitude (from $\varepsilon_N \cong 0.01$m-rad to ~0.00005m-rad). The longitudinal emittance is reduced by ~ one order of magnitude, although it has been reduced by more than two orders of magnitude before the final emittance exchanges.

In this section we describe several characteristic emittance exchange steps. These examples will illustrate the cooling parameters as well as demonstrate the use of the present exchange model in designing a matched and optimized cooling system.

The first example we consider corresponds to beam conditions near the beginning of the system, where the energy spread and emittances are both quite large. In this case we choose μ-beam at an initial kinetic energy of 300 MeV (momentum of 391.7 MeV), an initial rms momentum spread δ_1 of 7.4%, and rms normalized emittance of 0.015m-rad (geometric emittance of $\varepsilon_\perp = 0.004$). The beam is focussed onto a beryllium wedge absorber (dE/ds = 3 MeV/cm) with $\beta_1 = 0.34$m ($\sigma = 3.7$cm) at a dispersion of 1m, so the ratio of momentum to emittance beam size is 2. The example is matched to obtain zero dispersion after the wedge (case C), which implies $\tan\theta = 1$. The wedge is therefore designed to reduce the energy spread by $\sqrt{5}$ while increasing transverse emittance by the same factor.

This first example has also been simulated using SIMUCOOL.[10,11] Even though this is a wedge case with very large energy spread and large emittance, results in good agreement with the linear model were obtained. The wedge must be thick enough to accommodate the full momentum spread as well as the full beam size, which means that cooling and scattering (rms heating terms) are nonnegligible. With a wedge with a 17 cm thickness at the beam center; the mean energy decreases ~50 MeV in energy to 340 MeV/c. Without a reoptimization, the dispersion was reduced from 1 m to <0.05 m and an exchange of a factor of ~2 has been obtained (rather than $\sqrt{5} = 2.236$), accompanied by ~10% emittance cooling.

The second case we consider corresponds to conditions near the middle of the cooling sequence. The beam momentum is 200 MeV/c and a 0.035m Be wedge at $\eta = 0.5$m and $\beta = 0.15$m reduces the momentum spread from 4% to 2.5% while emittance is increased by a factor of 1.6. This case is also matched to zero dispersion exiting the wedge, and the exit of the wedge would be an appropriate place to put an extended absorber for transverse cooling (possibly a Li or Be focussing rod for extended cooling). SIMUCOOL results are in good agreement with this general process.

The third case we consider is one from near the end of the cooling sequence, where the longitudinal phase space is increased in order to reduce the transverse emittance (see Fig. 3), in order to obtain minimal final emittances for the μ^+-μ^- collider. This wedge is arranged so as to increase the energy spread, and that condition is obtained by choosing $\tan \theta < 0$, or alternatively a negative dispersion. The beam energy is 25 MeV with a normalized emittance of 61 mm-mrad and initial $\delta = 0.0081$, and the beam is focussed to small β^* (0.014m) at small dispersion (0.0105m) with a 0.0017 m thick wedge. We expect a decrease of emittance by a factor of 1.6 with a corresponding increase of $\delta p/p$. SIMUCOOL results are also in reasonable agreement with this simplified model.[11]

Discussion

As shown in the above examples, typical wedges used for phase-space exchange are relatively large, and a large exchange formalism is useful. Transformations through a wedge significantly change emittances, betatron functions and dispersion. Matching of these changes is posssible in the present formalism. Particular examples, such as dispersion-suppressing wedges, will be useful in generating compact cooling scenarios.

References

1. D. Neuffer and R. Palmer, Proc. European Particle Accelerator Conference EPAC94, London, p.52, 1994.

2. J. Gallardo, ed. Proc. 9th Advanced ICFA Beam Dynamics Workshop, "Beam Dynamics and Technology Issues for μ^+-μ^- Colliders", Montauk NY 1995, AIP Conf. Proc. 372 (1996).

3. R. B. Palmer et al, ibid, AIP Conf. Proc. 372, 3 (1996).

4. D. Neuffer, Particle Accelerators, 14, 75 (1983).

5. D. Neuffer, Nucl Inst. and Meth. A350, 27 (1994).

6. A. N. Skrinsky and V. V. Parkhomchuk, Sov. J. Nucl. Phys. 12, 3 (1981).

7. M. Sands, SLAC-121 (1970), also in Proc. of the Int. School of Physics, B. Touschek, ed. Varenna 1969, Academic Press NY (1971).

8. E. Courant and H. L. Snyder, Annals of Physics 3,1 (1957).

9. J. M. Peterson, Proc. 1983 Particle Accelerator Conference, IEEE Trans. NS-30, 2403 (1983).

10. A. Van Ginneken, Nucl. Inst. and Meth. A 362, 213 (1995).

11. A. Van Ginneken and D. Neuffer, studies in progress (1996).

Table 1: Parameter list for a 4 TeV μ^+-μ^- Collider

Parameter	Symbol	Value
Energy per beam	E_μ	2 TeV
Luminosity	$L = f_0 n_s n_b N_\mu^2 / 4\pi\sigma^2$	10^{35} cm^{-2}s^{-1}
Source Parameters		
Proton energy	E_p	30 GeV
Protons/pulse	N_p	$2 \times 3 \times 10^{13}$
Pulse rate	f_0	15 Hz
μ-production acceptance	μ/p	.2
μ-survival allowance	N_μ/N_{source}	.33
Collider Parameters		
Number of μ /bunch	$N_{\mu\pm}$	2×10^{12}
Number of bunches	n_B	1
Storage turns	$2n_s$	2000
Normalized emittance	ε_N	3×10^{-5} m-rad
μ-beam emittance	$\varepsilon_t = \varepsilon_N/\gamma$	1.5×10^{-9} m-rad
Interaction focus	β_0	0.3 cm
Beam size at interaction	$\sigma = (\varepsilon_t\beta_0)^{\frac{1}{2}}$	2.1 μm

Figure 1: Overview of the beam transformation in passing through a wedge absorber. The upper portion shows a stylized view of a beam passing through a dispersive transport into a wedge absorber; the lower portion shows the projection of the 6-D beam phase space ellipse into x-δ phase space, and its changes passing through the system. Dispersion imposes an x-δ correlation (ellipse tilt), and the wedge reduces the beam energy $\delta(x)$, with energy loss a function of x: $\Delta\delta = x\, dp/ds\, \tan\theta/p$. Note that the x-$\delta$ ellipse area remains the same (in the limit where average energy loss is zero).
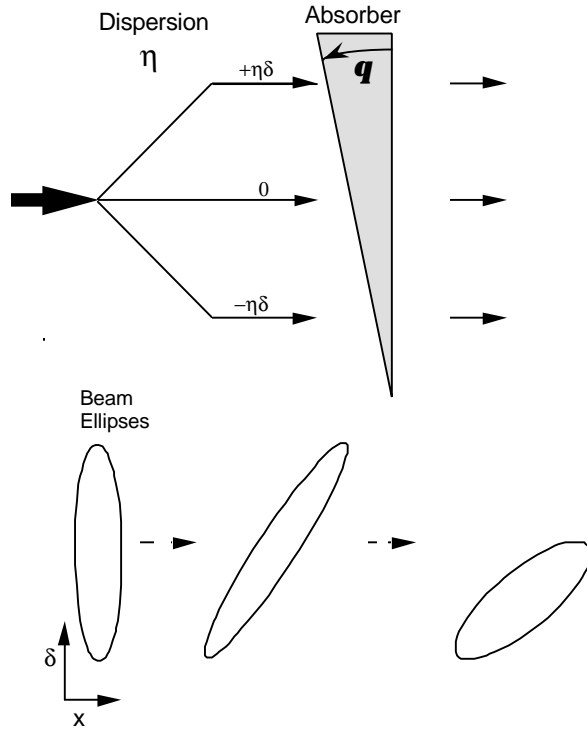
Figure 2: equivalent wedge absorbers, tilted and symmetric cases.

Absorber   Absorber

q f

$+\eta\delta$

0

$-\eta\delta$

tilted   **symmetric**
absorber   **absorber**
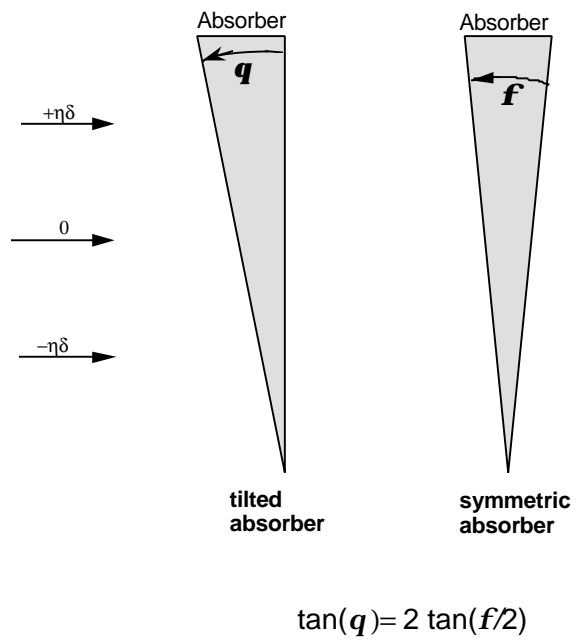
$$\tan(q) = 2\tan(f/2)$$

Figure 3. Transformation of phase space ellipses where the wedge is designed to increase energy spread; the wedge is oriented so that lower energy particles go through the thicker end of the wedge.
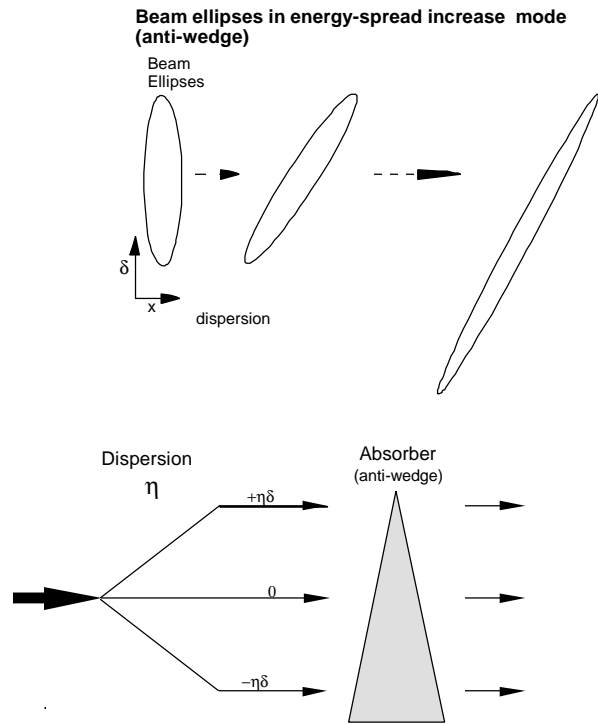
Beam ellipses in energy-spread increase mode (anti-wedge)

Beam
Ellipses

δ

x

dispersion

Dispersion
η

$+\eta\delta$

0

$-\eta\delta$

Absorber
(anti-wedge)

Table 2: Examples of Phase Space Exchange

Exchange Parameter	Example 1	Example 2	Example 3
E_{beam}	300	120	25
P_{beam}	392	199.4	76.9
$\delta_0 = \delta P/P$ (initial)	0.074	0.04	0.0081
η_0 initial dispersion	1.0m	0.5	-0.105
σ_0 - initial beam size	0.037m	0.017	0.001
β_0 - initial betatron function	0.34m	0.15	0.013
ε_0 - initial transverse emittance (unnormalized)	0.004 m-rad	0.00212	84×10^{-6}
Wedge material	Be	Be	LiH
dp/ds (MeV/c/cm)	3.0	3.4	5.70
$\tan \theta$	1.0	0.65	0.6
$\delta' = $ dp/ds $\tan \theta/p$	0.80	1.107	0.044
Exchange factor	0.44	0.67	1.60
thickness (2σ)	0.17	0.035	0.0017